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Low-Thrust Maneuvers Near the Libration Points

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The minimum propellant optimal maneuvers of space vehicles equipped with low-thrust propulsion installation near the libration points in the Earth-moon system are examined. The variational problem for determining the optimal laws that characterize these maneuvers is reduced to a problem of the type previously formulated by the author. On the basis of the optimal laws established in this manner, numerical applications are carried out.

Nomenclature

a_ξ, a_η	= components of acceleration due to thrust upon the axes of the system $L\xi\eta$
D	= distance between Earth and moon
g	= Earth gravitational acceleration
Gxy	= rotating system of coordinate axes with the origin at the barycenter G of the Earth-moon system (Fig. 1)
$L\xi\eta$	= rotating system of coordinate axes with the origin at the libration point (Fig. 1)
m, m_1, m_2	= mass of vehicle, Earth, and moon, respectively
r	= distance from Earth to the libration point
S_1	= distance between barycenter and Earth
S_2	= distance between barycenter and moon
t	= time
V_ξ, V_η	= velocity components of the vehicle upon the axes of the system $L\xi\eta$
$\bar{x}, \bar{y}$	= coordinates of the libration point in the system Gxy
ζ	= distance from moon to the libration point
μ	= Earth gravitational parameter
ξ, η	= coordinates of the vehicle in the system of axes $L\xi\eta$
τ	= combustion (maneuver) duration
ω	= angular velocity about the barycenter

Introduction

THE examination of transfer maneuvers from the libration points of the Earth-moon system toward orbits of the Earth or moon, and conversely, has been the object recently of some remarkable papers.[1-6] The impulse technique described in Ref. 2 is useful for the transfer from the libration points toward orbits of the Earth or moon. For the transfer from Earth or moon orbits to libration points, this technique leads to some difficulties. It is well known that, in the transfer from Earth orbits to the libration points, the initial velocities differ slightly, and small errors in these velocities or the influence of some perturbations can lead to great deviations of the vehicle from the target. In this respect, Ref. 1 brings a contribution, but the problems remain open. The impulse technique certainly can bring the vehicle on orbits around the libration points or close to them. The question that arises is, by what means can the vehicle arrive in such cases at the libration points? A first investigation carried out in this paper can give an answer: the use of the technique of low-thrust, which, in addition, can bring the vehicle from the libration points near to or into orbits around these points.

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Motion Equations

Let us assume that a vehicle equipped with a low-thrust propulsion installation is near a libration point L (Fig. 1). We consider the planar motion in the restricted frame of the three bodies first in the rotating system Gxy (Fig. 1), where we take $D = S_1 + S_2 = 1$ and the time unit chosen such that $\mu = 1$. The unit of time is the period of the moon's orbit divided by 2π and multiplied by the square root of the quantity one plus the moon/Earth mass ratio, and the unit of mass is the Earth's mass. The vehicle is considered as an infinitesimal body. The Earth and moon masses fulfill the condition $m_1 \gg m_2$, and the perturbing influence of the Sun is neglected.

Then, by setting $\xi = x - \bar{x}$ and $\eta = y - \bar{y}$ in the system $L\xi\eta$ with the origin at the libration point,[7] the equations of the vehicle relative motion subject to the action of the propulsion installation can be written in the form

$$\frac{d\xi}{dt} = V_\xi \tag{1a}$$

$$\frac{dV_\xi}{dt} = 2\omega V_\eta + K_1 \xi + K_3 \eta + a_\xi \tag{1b}$$

$$\frac{d\eta}{dt} = V_\eta \tag{1c}$$

$$\frac{dV_\eta}{dt} = -2\omega V_\xi + K_2 \eta + K_3 \xi + a_\eta \tag{1d}$$

where, for the equidistant libration points,[7]

$$K_1 = \tfrac{3}{4}(1 + m^*) \tag{2a}$$

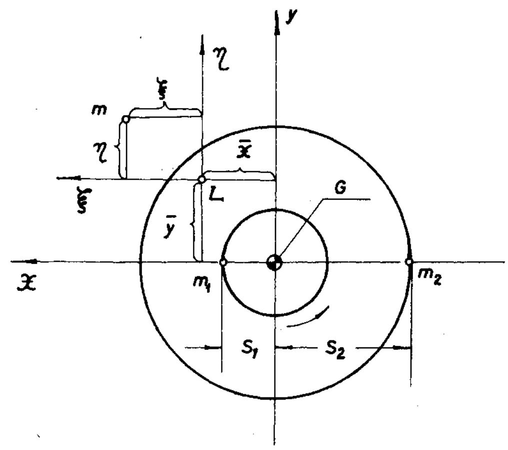

Fig. 1 Schematic presentation of vehicle motion near the libration points in the Earth-moon system.

$$K_2=(9/4)(1+m^*) \tag{2b}$$

$$K_3=\mp \tfrac{3}{4}\sqrt{3}(1-m^*) \tag{2c}$$

and, for collinear libration points,[7]

$$K_1=1+m^*+(2/r^3)+(2m^*/\zeta^3) \tag{3a}$$

$$K_2=1+m^*-(1/r^3)-(m^*/\zeta^3) \tag{3b}$$

$$K_3=0 \ (m^*=m_2/m_1) \tag{3c}$$

Variational Problem

Under the assumption that, in the maneuvers mentioned previously, around the libration points, the parameters of the motion at the beginning and end of these maneuvers are known, and there are no constraints on the control, the variational problem with contraints, with fixed extremities, reduces to finding the minimum of the functional

$$J=\int_0^T (a_\xi^2+a_\eta^2)\mathrm{d}t \tag{4}$$

with the conditions

$$\phi_1=\frac{\mathrm{d}\xi}{\mathrm{d}t}-V_\xi=0 \tag{5a}$$

$$\phi_2=\frac{\mathrm{d}V_\xi}{\mathrm{d}t}-2\omega V_\eta-K_1\xi-K_3\eta-a_\xi=0 \tag{5b}$$

$$\phi_3=\frac{\mathrm{d}\eta}{\mathrm{d}t}-V_\eta=0 \tag{5c}$$

$$\phi_4=\frac{\mathrm{d}V_\eta}{\mathrm{d}t}+2\omega V_\xi-K_2\eta-K_3\xi-a_\eta=0 \tag{5d}$$

By reducing the extremum problem with constraints to an extremum problem without constraints, we introduce Lagrange's multipliers $\lambda_1, \lambda_2,...,\lambda_4$ and form the function

$$F=a_\xi^2+a_\eta^2+\lambda_1\phi_1+\lambda_2\phi_2+\lambda_3\phi_3+\lambda_4\phi_4 \tag{6}$$

by means of which Euler's equations

$$\frac{\partial F}{\partial \xi}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial(\mathrm{d}\xi/\mathrm{d}t)}=0 \tag{7a}$$

$$\vdots$$

$$\frac{\partial F}{\partial a_\eta}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial(\mathrm{d}a_\eta/\mathrm{d}t)}=0 \tag{7b}$$

lead to the following differential equations:

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=-\lambda_2K_1-\lambda_4K_3 \tag{8a}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=2\omega\lambda_4-\lambda_1 \tag{8b}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}t}=-\lambda_2K_3-\lambda_4K_2 \tag{8c}$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}t}=-\lambda_3-2\omega\lambda_2 \tag{8d}$$

and to the algebraic equations:

$$2a_\xi-\lambda_2=0 \qquad 2a_\eta-\lambda_4=0 \tag{9}$$

By integrating the differential equations of the extremals (1) and (8) and taking account of Eq. (9), we obtain the desired extremals $\xi(t)$, $\eta(t)$, $V_\xi(t)$, $V_\eta(t)$, $a_\xi(t)$, $a_\eta(t)$, which characterize the minimum propellant optimal maneuvers near the libration points.

Approximate Analytical Solution for the Collinear Libration Points

In agreement with Refs. 1 and 2, which foresee the possibility of space operations related to the collinear libration points, in the following we establish approximate analytical solutions for studying optimal maneuvers around these points. Setting $K_3=0$ in Eqs. (1) and (8), we obtain for the collinear libration points the following system of differential equations of the extremals:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t}=V_\xi \tag{10a}$$

$$\frac{\mathrm{d}V_\xi}{\mathrm{d}t}=2\omega V_\eta+K_1\xi+a_\xi \tag{10b}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t}=V_\eta \tag{10c}$$

$$\frac{\mathrm{d}V_\eta}{\mathrm{d}t}=-2\omega V_\xi+K_2\eta+a_\eta \tag{10d}$$

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t}=-K_1\lambda_2 \tag{10e}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t}=2\omega\lambda_4-\lambda_1 \tag{10f}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}t}=-K_2\lambda_4 \tag{10g}$$

$$\frac{\mathrm{d}\lambda_4}{\mathrm{d}t}=-\lambda_3-2\omega\lambda_2 \tag{10h}$$

System (10) can be integrated exactly by ordinary methods. The rigorous analytical solutions that are obtained have, however, very complicated expressions.

Therefore, neglecting the Coriolis accelerations, taking into account that in the Earth-moon system for the units adopted in the motion equations $K_1>0$ and $K_2<0$, setting $k=\sqrt{K_1}$, $K_2=-K_2'$, and $\kappa=\sqrt{K_2'}$, the system of differential equations of the extremals can be put in the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2}-k^2\xi=a_\xi \tag{11a}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}t^2}+\kappa^2\eta=a_\eta \tag{11b}$$

$$\frac{\mathrm{d}^2\lambda_2}{\mathrm{d}t^2}-k^2\lambda_2=0 \tag{11c}$$

$$\frac{\mathrm{d}^2\lambda_4}{\mathrm{d}t^2}+\kappa^2\lambda_4=0 \tag{11d}$$

Computations showed that neglecting the Coriolis accelerations does not introduce large errors, especially for a_ξ and a_η. The simplification introduced by this approximation is essential, as it leads to very simple approximate analytical solutions. Indeed, by integrating Eqs. (11c) and (11d) and taking account of the algebraic equations (9), we obtain†

$$a_\xi=\tfrac{1}{2}(C_1\cosh kt+C_2\sinh kt) \tag{12a}$$

$$\alpha_\eta=\tfrac{1}{2}(C_3\cos\kappa t+C_4\sin\kappa t) \tag{12b}$$

†Equation (12a) is obtained by writing $a_\xi=\tfrac{1}{2}(C_1'e^{kt}+C_2'e^{-kt})$ and expressing the e^{kt}, e^{-kt} function of cosh kt, sinh kt.

Inserting Eqs. (12) in Eqs. (11a) and (11b) and integrating, we obtain

$$\xi = [C_5 + (1/4k)C_2 t]\cosh kt + [C_6 + (1/4k)C_1 t]\sinh kt \tag{13a}$$

$$\eta = [C_7 - (1/4\kappa)C_4 t]\cos\kappa t + [C_8 + (1/4\kappa)C_3 t]\sin\kappa t \tag{13b}$$

from which

$$V_\xi = [C_6 k + (1/4k)C_2 + \tfrac{1}{4}C_1 t]\cosh kt + [C_5 k + (1/4k)C_1 + \tfrac{1}{4}C_2 t]\sinh kt \tag{14a}$$

$$V_\eta = [C_8 \kappa - (1/4\kappa)C_4 + \tfrac{1}{4}C_3 t]\cos\kappa t + [-C_7\kappa + (1/4\kappa)C_3 + \tfrac{1}{4}C_4 t]\sin\kappa t \tag{14b}$$

The sought for functions $\xi(t)$, $\eta(t)$, ..., $a_\eta(t)$, which characterize the optimal maneuvers around the collinear libration points, contain eight arbitrary constants C_1, C_2, ..., C_8. For the descent from orbits at the libration points or for their approach from neighboring points, the problem becomes similar to that of orbital rendezvous.[8,9]

From the initial conditions ($t=0$),

$$\xi(0) = \xi_0 \qquad V_\xi(0) = V_{\xi_0} \tag{15a}$$

$$\eta(0) = \eta_0 \qquad V_\eta(0) = V_{\eta_0} \tag{15b}$$

and from the final conditions ($t=\tau$),

$$\xi(\tau) = 0 \qquad V_\xi(\tau) = 0 \tag{16a}$$

$$\eta(\tau) = 0 \qquad V_\eta(\tau) = 0 \tag{16b}$$

we obtain the following expressions for the constants C_1, C_2, ..., C_8:

$$C_1 = 4k\frac{k\sinh^2 k\tau\,\xi_0 - (k\tau - \sinh k\tau\cosh k\tau)V_{\xi_0}}{(k\tau)^2 - \sinh^2 k\tau} \tag{17a}$$

$$C_2 = -4k\frac{k(k\tau + \sinh k\tau\cosh k\tau)\xi_0 + \sinh^2 k\tau\,V_{\xi_0}}{(k\tau)^2 - \sinh^2 k\tau} \tag{17b}$$

$$C_3 = -4\kappa\frac{\kappa\sin^2 \kappa\tau\,\eta_0 - (\kappa\tau - \sin\kappa\tau\cos\kappa\tau)V_{\eta_0}}{(\kappa\tau)^2 - \sin^2 \kappa\tau} \tag{17c}$$

$$C_4 = 4\kappa\frac{\kappa(\kappa\tau + \sin\kappa\tau\cos\kappa\tau)\eta_0 + \sin^2 \kappa\tau\,V_{\eta_0}}{(\kappa\tau)^2 - \sin^2 \kappa\tau} \tag{17d}$$

$$C_5 = \xi_0 \tag{17e}$$

$$C_6 = \frac{(k\tau + \sinh k\tau\cosh k\tau)\xi_0 + k\tau^2 V_{\xi_0}}{(k\tau)^2 - \sinh^2 k\tau} \tag{17f}$$

$$C_7 = \eta_0 \tag{17g}$$

$$C_8 = \frac{(\kappa\tau + \sin\kappa\tau\cos\kappa\tau)\eta_0 + \kappa\tau^2 V_{\eta_0}}{(\kappa\tau)^2 - \sin^2 \kappa\tau} \tag{17h}$$

By means of the approximate solutions, Eqs. (12-14), a series of numerical applications was carried out for the optimal maneuvers of nearing and reaching the libration point L_2 in the Earth-moon system, for which we have

$$m^* = m_2/m_1 = 0.01227 \qquad D = 3.84\times 10^8\ \text{m}$$

$$\mu = 3.987\times 10^{14}\ \text{m}^3/\text{s}^2$$

$$r = 1.1682D \qquad \zeta = 0.1682D$$

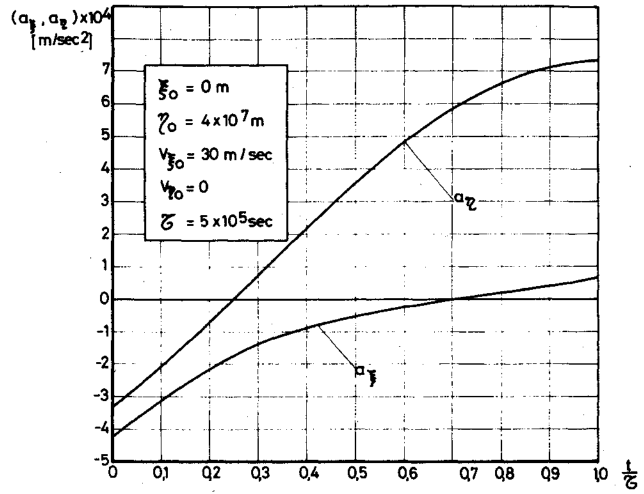

Fig. 2 Components of the acceleration due to thrust in the $L\xi\eta$ system.

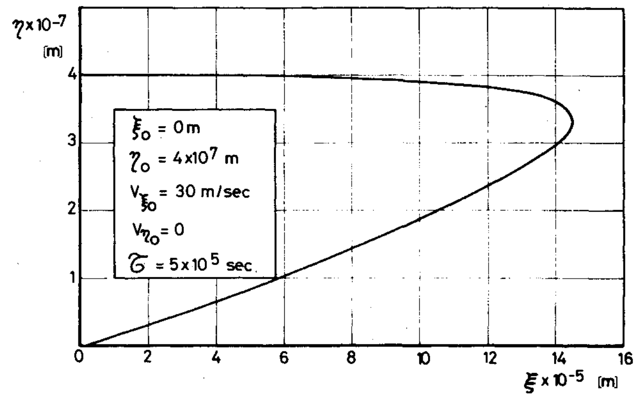

Fig. 3 Optimal trajectory of the vehicle.

By means of the initial data

$$\xi_0 = 0\ \text{m} \qquad V_{\xi_0} = 30\ \text{m/s}$$

$$\eta_0 = 40\times 10^6\ \text{m} \qquad V_{\eta_0} = 0\ \text{m/s}$$

which correspond approximately to the position of the vehicle at the apogee of an undisturbed elliptical orbit around the point L_2 having the characteristics major semiaxis $a = 40\times 10^6$ m, eccentricity $\epsilon = 0.9518$, and period of revolution $T = 1.329\times 10^6$ s for a combustion duration of $\tau = 5\times 10^5$ s, we have calculated the components of acceleration due to thrust a_ξ, a_η as a function of t/τ (Fig. 2) and the coordinates $\xi(t)$, $\eta(t)$ from which the optimal trajectory $\eta = \eta(\xi)$ has been deduced (Fig. 3).

The same quantities also have been calculated for combustion durations of $\tau = 10^5$ and 10^4 s. Another numerical application has been carried out with the initial data

$$\xi_0 = -10^6\ \text{m} \qquad \eta_0 = 5\times 10^4\ \text{m}$$

$$V_{\xi_0} = 5\text{-}100\ \text{m/s} \qquad V_{\eta_0} = (-0.25)\text{-}(-10)\ \text{m/s}$$

for maneuver durations of $\tau = 10^4$ and 10^5 s, where the shape of the optimal trajectories are, in general, close to straight lines. We stress that solutions (12) and (13) were calculated in the units $D = 1$ and the time chosen such that $\mu = 1$; the results then were converted into units of meters and seconds.

Conclusions

Preliminary calculations carried out in the present paper show that, around the libration points, maneuvers of vehicles with low-thrust propulsion are possible, and that their domain of application can be established. Some numerical examples show that, for durations of the maneuvers of $\geq 10^5$ s (compatible with the low-thrust assumption), the value of the acceleration due to thrust falls within the allowable limits 10^{-3} g-$10^{-6}g$. We stress that the upper limit $10^{-3}g$ for the acceleration due to thrust is mentioned in the literature[10] as presently feasible.

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